Abstract

Hefetz, Mütze, and Schwartz conjectured that every connected undirected graph admits an antimagic orientation [11]. In this paper we support the analogous question for distance magic labeling. Let $\Gamma$ be an Abelian group of order $n$. A directed $\Gamma$-distance magic labeling of an oriented graph $\vec{G} = (V,A)$ of order $n$ is a bijection $\vec{l} : V \rightarrow \Gamma$ with the property that there is a magic constant $\mu \in \Gamma$ such that for every $x \in V(G)$ $w(x) = \sum_{y \in N^+(x)} \vec{l}(y) - \sum_{y \in N^-(x)} \vec{l}(y) = \mu$. In this paper we provide an infinite family of odd regular graphs possessing an orientable $\mathbb{Z}_n$-distance magic labeling. Our results refer to lexicographic product of graphs. We also present a family of odd regular graphs that are not orientable $\mathbb{Z}_n$-distance magic.

1 Introduction

Consider a simple graph $G$ and a simple oriented graph $\vec{G}$. We denote by $V(G)$ the vertex set and by $E(G)$ the edge set of $G$. For $\vec{G}$ we denote by $V(\vec{G})$ the vertex set and by $A(\vec{G})$ the arc set of $\vec{G}$. We denote the order of $G$ by $|V(G)|$. An arc $\vec{xy}$ is considered to be directed from $x$ to $y$, moreover $y$ is called the head and $x$ is called the tail of the arc. For a vertex $x$, the set of head endpoints adjacent to $x$ is denoted by $N^-(x)$, and the set of tail endpoints adjacent to $x$ is denoted by $N^+(x)$. For graph theory notations and terminology not described in this paper, the readers are referred to [9].

In this paper we investigate distance magic labelings, which belong to a large family of magic-type labelings. Probably the best known problem
in the area of magic and antimagic labelings is the antimagic conjecture by Hartsfield and Ringel [10], which claims that the edges of every graph except $K_2$ can be labeled by integers $1, 2, \ldots, |E|$ so that the weight of each vertex is different. The conjecture is still open. Twenty years later Heftz, Mütze, and Schwartz introduced the variation of antimagic labelings, i.e., antimagic labelings on directed graphs. Moreover, they conjectured that every connected undirected graph admits an antimagic orientation [11]. The papers [4, 5, 6] stated the analogous question for distance magic labeling, namely when a graph $G$ of order $n$ has a $\mathbb{Z}_n$-distance magic orientation.

Froncek in [7] defined the notion of group distance magic graphs, i.e., the graphs allowing a bijective labeling of vertices with elements of an Abelian group resulting in constant sums of neighbour labels. Some graphs which are group distance magic can be seen for example in [1, 2, 3, 7].

An orientable $\mathbb{Z}_n$-distance magic labeling of a graph, first introduced by Cichacz, Freyberg and Froncek [4], is a generalization of group distance magic labeling. Let $\Gamma$ be an Abelian group of order $n$. Formally speaking, a directed $\Gamma$-distance magic labeling of an oriented graph $\vec{G} = (V, A)$ of order $n$ is a bijection $\vec{l} : V \to \Gamma$ with the property that there is a magic constant $\mu \in \Gamma$ such that for every $x \in V(G)$

$$w(x) = \sum_{y \in N^+(x)} \vec{l}(y) - \sum_{y \in N^-(x)} \vec{l}(y) = \mu,$$

If for a graph $G$ there exists an orientation of $\vec{G}$ such that there is a directed $\Gamma$-distance magic labeling $\vec{l}$ for $\vec{G}$, we say that $G$ is orientable $\Gamma$-distance magic and the directed $\Gamma$-distance magic labeling $\vec{l}$ we call an orientable $\Gamma$-distance magic labeling.

Cichacz, Freyberg and Froncek proved the following theorem.

**Theorem 1** ([4]). Let $G$ be an $r$-regular graph on $n \equiv 2 \pmod{4}$ vertices, where $r$ is odd. There does not exist an orientable $\mathbb{Z}_n$-distance magic labeling for the graph $G$.

Moreover they showed the following.

**Theorem 2** ([4]). The complete graph $K_n$ is orientable $\mathbb{Z}_n$-distance magic if and only if $n$ is odd.

**Theorem 3** ([4]). Let $G = K_{n_1, n_2, \ldots, n_k}$ be a complete $k$-partite graph such that $n_1 \leq n_2 \leq \ldots \leq n_k$ and $n = n_1 + n_2 + \ldots + n_k$ is odd. The graph $G$ is orientable $\mathbb{Z}_n$-distance magic graph if $n_2 \geq 2$. 
In this paper we consider two out of four standard graph products (see [8]). The Cartesian product \( G \square H \) of graphs \( G \) and \( H \) is a graph such that the vertex set of \( G \square H \) is the Cartesian product \( V(G) \times V(H) \) and any two vertices \((g, h)\) and \((g', h')\) are adjacent in \( G \square H \) if and only if either \( g = g' \) and \( h \) is adjacent with \( h' \) in \( H \) or \( h = h' \) and \( g \) is adjacent with \( g' \) in \( G \).

The lexicographic product \( G \circ H \) is a graph with the vertex set \( V(G) \times V(H) \). Two vertices \((g, h)\) and \((g', h')\) are adjacent in \( G \circ H \) if and only if either \( g \) is adjacent with \( g' \) in \( G \) or \( g = g' \) and \( h \) is adjacent to \( h' \) in \( H \). \( G \circ H \) is also called the composition of graphs \( G \) and \( H \) and denoted by \( G[H] \) (see [9]).

So far there was known only one example of odd regular graph orientable \( Z_n \)-distance magic ([4]). In Section 2 we give necessary and sufficient conditions for \( K_m \circ K_n \sim \) being orientable \( Z_{mn} \)-distance magic. As a consequence, we provide an infinite family of odd regular graphs possessing an orientable \( Z_n \)-distance magic labeling. In the last section we present some family of odd regular graphs that are not orientable \( Z_n \)-distance magic.

2 Lexicographic products

Consider a graph \( G = K_m \circ K_n \). Let us denote independent sets of vertices by \( V_1, V_2, \ldots, V_m \) where \( V_i = \{v_{i1}, v_{i2}, \ldots, v_{in}\} \) for \( i \in \{1, 2, \ldots, m\} \). Now we will give necessary and sufficient conditions for \( K_m \circ K_n \) being orientable \( Z_{mn} \)-distance magic. The following theorem will be useful in our proof.

**Theorem 4** ([12]). Let \( n = r_1 + r_2 + \ldots + r_q \) be a partition of the positive even integer \( n \), where \( r_i \geq 2 \) for \( i = 1, 2, \ldots, q \). Let \( A = \{-\frac{n}{2}, -(\frac{n}{2} - 1), \ldots, -1, 1, \ldots, (\frac{n}{2} - 1), \frac{n}{2}\} \). Then the set \( A \) can be partitioned into pairwise disjoint subsets \( A_1, A_2, \ldots, A_q \) such that for every \( 1 \leq i \leq q \), \( |A_i| = r_i \) with \( \sum_{a \in A_i} a = 0 \).

**Theorem 5.** A graph \( G = K_m \circ K_n \) is orientable \( Z_{mn} \)-distance magic if and only if \( n \not\equiv 1 \pmod{2} \) or \( m \not\equiv 2 \pmod{4} \) for \( n \geq 2 \). When \( n = 1 \), \( G \) is orientable \( Z_{mn} \)-distance magic if and only if \( m \) is odd.

*Proof.** By Theorem 2 we can assume that \( n > 1 \). If \( mn \) is odd, then we are done by Theorem 3. Moreover from Theorem 1 we can conclude that for \( n \equiv 1 \pmod{2} \) and \( m \equiv 2 \pmod{4} \) there does not exist an orientable \( Z_{mn} \)-distance magic labeling for the graph \( G \). We will consider now two cases:
Case 1: $mn \equiv 0 \pmod{4}$.

Let $A = \left\{ -\frac{mn}{2} + 1, -\frac{mn}{2} + 2, \ldots, -1, \ldots, \frac{mn}{2} - 2, \frac{mn}{2} - 1 \right\} = \mathbb{Z}_{mn} \setminus \{0, \frac{mn}{2}\}$.

If $n = 2$, then let $A^1 = \{0, \frac{mn}{2}\}$ and there exists a zero-sum partition $A^2, A^3, \ldots, A^m$ of the set $A$ such that $|A^i| = 2$ for every $2 \leq i \leq m$ by Theorem 4. Let $q$ be the index of subset containing $\frac{mn}{4}$.

If $n > 2$, then by Theorem 4 there exists a partition of $A$ into $A^1, A^2, \ldots, A^m$ such that $|A^1| = |A^2| = n - 1, |A^i| = n$ for $i = 3, 4, \ldots, m$ and $\sum_{a \in A^i} a = 0$ for $i = 1, 2, \ldots, m$. Without loss of generality $\frac{mn}{4} \in A_q$, where $q \neq 1$. Let $A^1 = A^1_q \cup \{\frac{mn}{2}\}, A^2 = A^2_q \cup \{0\}, A^i = A^i_q$, where $i = 3, 4, \ldots, m$.

Note that in both situations $\sum_{a \in A^i} a \equiv 0 \pmod{mn}$ for $i = 2, 3, \ldots, m$. Label vertices from each set $V^i$ by the elements of $A^i$ with the restriction that $\overrightarrow{uv}(v^1_q) = \frac{mn}{2}$ and $\overrightarrow{uv}(v^q_j) = \frac{mn}{4}$.

Let $o(uv)$ be the orientation for the edge $uv$. For edges $v^1_qv^q_j$ from $E(G)$ we have

$$o(v^1_qv^q_j) = \begin{cases} \overrightarrow{v^1_qv^q_j}, & i \in \{1, 2, \ldots, n\}, j = 1 \\ \overrightarrow{v^q_jv^1_q}, & i \in \{1, 2, \ldots, n\}, j \in \{2, 3, \ldots, n\}. \end{cases}$$

This way we obtained the edge orientation between $V^1$ and $V^q$. For the remaining pairs of partition vertex sets $V^k$ and $V^l$ we demand all edges to be oriented the same from the $k$-th set to the $l$-th set or conversely. One can check that $w(v) \equiv \frac{mn}{4} \pmod{mn}$ for any $v \in V(G)$. Thus $G$ is orientable $\mathbb{Z}_{mn}$-distance magic.

Case 2: $m \equiv 1 \pmod{4}$ and $n \equiv 2 \pmod{4}$.

For the set $V^k$ we introduce the following orientation

$$o(v^k_lv^l_j) = \overrightarrow{v^k_lv^l_j}$$

for all $l \in \{k - \frac{m-1}{2}, k - \frac{m-1}{2} + 1, \ldots, k - 1\}$ (where operations are taken modulo $m$). For the remaining edges we have

$$o(v^k_lv^l_j) = \overrightarrow{v^k_lv^l_j}.$$

We say that $V^l$ precedes $V^k$ and $V^k$ succeeds $V^l$ if arcs between vertices of $V^l$ and $V^k$ have tails in $V^l$ and heads in $V^k$. Define the labeling $\overrightarrow{v}$ such that $\overrightarrow{v}(v^k_i) = (i - 1) + (k - 1)n$, where $i \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots, m\}$. It is easy to see that for each $k \in \{1, 2, \ldots, m\}, i \in \{1, 2, \ldots, n\}$ we have

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$w(v_i^k) = \sum_{v \in V^i; V^i \text{ preceds } V^k} \overline{l}(v) - \sum_{v \in V^i; V^i \text{ succeeds } V^k} \overline{l}(v) = \frac{m-1}{2} nd \pmod{mn},$

where $d$ is a constant difference between labels $a_i^j \in A^j$ and $a_i^{j'} \in A^{j'}$ where $j' = j + \frac{m+1}{2} \pmod{m}$. Therefore $d = n\frac{m+1}{2}$. We obtain the magic constant $\mu \equiv \frac{m-1}{2} \frac{m+1}{2} n^2 \pmod{mn}$.

Observe that if $G$ is an odd regular graph, then the lexicographic product $G \circ \overline{K}_{2n+1}$ is also an odd regular graph. From the above Theorem 5 we obtain the following observation showing that there exist infinitely many odd regular graphs that are orientable $\Gamma$-distance magic for a cyclic group $\Gamma$.

**Observation 1.** The lexicographic product $K_{4m} \circ \overline{K}_{2n+1}$ has a $\mathbb{Z}_{4m(2n+1)}$-distance magic labeling for any $m, n \geq 1$.

Note that the method presented above works also for other families of graphs, for instance for $K_{1,4m+3} \circ \overline{K}_n$. We just assign the label $\frac{4(m+1)n}{2}$ to some vertex in the center of the $K_{1,4m+3} \circ \overline{K}_n$ and place $\frac{4(m+1)n}{4}$ in the other set. The orientation in $K_{1,4m+3} \circ \overline{K}_n$ is similar to the general case.

### 3 Prism graphs

In this section we present some odd regular graphs that are not orientable distance magic. The *prism* is a Cartesian product $P_2 \square C_n$.

**Theorem 6.** Let us consider a prism graph $G$ of order $2n$. There does not exist an orientable $\mathbb{Z}_{2n}$-distance magic labeling.

**Proof.** If $|G| = |P_2 \square C_n| \equiv 2 \pmod{4}$ the thesis is fulfilled on the basis of Theorem 1. We are going to consider situation when $|G| \equiv 0 \pmod{4}$. Suppose that there exists an oriented $\mathbb{Z}_{2n}$-distance magic labeling for a graph $G$. Since $G$ is bipartite graph we can assume that it has partition sets $U = \{u_1, u_2, \ldots, u_{2k}\}$ and $W = \{w_1, w_2, \ldots, w_{2k}\}$.
Let us focus on the parity of the magic constant. There is no need to consider direction of edges because addition and subtraction modulo 2 give the same results. If we know the parity of three consecutive labels $\vec{l}(u_{i-1}), \vec{l}(u_i), \vec{l}(u_{i+1})$, then we can say what is the parity of the element $\vec{l}(u_{i+2})$. The parity of the magic constant generated by three consecutive labels needs to be preserved, which means:

$$(\vec{l}(u_i) + \vec{l}(u_{i+1}) + \vec{l}(u_{i+2})) \equiv (\vec{l}(u_{i+1}) + \vec{l}(u_{i+2}) + \vec{l}(u_{i+3})) \pmod{2}$$

Therefore, $\vec{l}(u_i) \equiv \vec{l}(u_{i+3}) \pmod{2}$ for any $i \in \{1, 2, \ldots, 2k\}$.

Hence knowing the parity of three initial labels one can establish parity of the remaining labels. We examine two possibilities, according to the cardinality of $U$.

Suppose first, that $|U| \not\equiv 0 \pmod{3}$. One can check that $\vec{l}(x) \equiv \vec{l}(y) \pmod{2}$ for any $x, y$ from the same partition set. Thus, $\vec{l}(u) \equiv 1 + \vec{l}(w) \pmod{2}$ for $u \in U$ and $w \in W$. Because the graph is odd regular the parity of the magic constant depends on the partition set, a contradiction.

We assume now that $|U| \equiv 0 \pmod{3}$, then we have to examine every three initial labels generating the rest of the sequence. For labels of the same parity we have contradiction that is compatible with the description above. If one of the three initial $\vec{l}(u_1), \vec{l}(u_2), \vec{l}(u_3)$ is an even number and the rest are odd numbers then in the whole $U$ component we have $\frac{1}{3}$ of all even numbers and $\frac{2}{3}$ of odd numbers. They generate even magic constant. On the other hand, in $W$ component we have $\frac{1}{3}$ of all odd numbers and $\frac{2}{3}$ of all even numbers (the rest of remaining labels). This does not allow a labeling that generates even magic constant because vertices from $W$ also should meet the rule of the same parity on every third element of the sequence. Therefore, there also should be $\frac{1}{3}$ of all even numbers and $\frac{2}{3}$ of odd numbers or all labels were even. That situation is not possible.

The case looks similar in the scheme with one odd number and two even numbers in initial three $\vec{l}(u_1), \vec{l}(u_2), \vec{l}(u_3)$. Above consideration exhausts other possible cases and therefore proves the rightness of the formula.
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References


